

An Integrated Scheme for Static Network Codes

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Abstract— In this paper, we analyze the inherent limitations of original definitions of static network codes. Based upon these analyses, we provide corresponding alternative definitions to overcome these limitations from a graph-theoretic perspective and prove them equivalent to the original one. After that, an integrated scheme for static network codes is proposed, under which all different classes of static network codes can be described, constructed and verified in an integrated way. As a result, we are able to have better position that affords an extensive overall view of static network codes in terms of linear independence.

Keywords— Network coding; Static network codes

I. INTRODUCTION

The seminal idea of network coding was first introduced by Yeung and Zhang [1] and then fully developed by Ahlswede *et al.* [2], where the maximum capacity of single-source multicast can be achieved by network coding rather than routing was proved. Shortly afterwards, the achievability of the max-flow bound by linear network codes was proved by Li *et al.* [3] using a vector space approach and then by Koetter and Médard [4] using a matrix approach. In [8], different classes of linear network codes are detailedly classified, namely generic, linear dispersion, linear broadcast and linear multicast. A concept of independent set was justified while a single-source network is being associated with a network matroid in [5]. Based on that, Min Tan *et al.* [6] propose a unified framework for these four linear network codes. However, previous theoretical work are based on the assumption that all the channels are available at all times and little research has been undertaken to investigate into the problem of linear network codes if links in network may fail. Koetter and Médard [4] first considered the problem of network recovery for nonergodic link failures and proved existence of static network codes that provide maximally robust networks for the multicast setup. Subsequently, Yeung *et al.* [7] described explicit definitions for static network codes.

In this paper, we first review the original definition of static network codes and analyse their inherent limitations. Then we extend results obtained in [6] from linear network codes to static network codes and propose an integrated scheme for static network codes. We will see later, under this scheme, all different classes of static network codes can be described, constructed and verified in an integrated way and the inherent limitations of original definitions of static network codes are overcome. Finally, we make use of the advantage of the integrated scheme to simplify some proofs that used to be complicated.

II. PRELIMINARIES

A communication network is modelled by a finite directed graph $G = (V, E)$, where V is the set of nodes and E is the set of channels in G which connect these nodes. Node without any incoming channels is called *source node*, denoted by S , and all other nodes are referred to as *non-source node*, denoted by T . We denote incoming channels of node i by $\text{In}(i)$ and outgoing channels of node i by $\text{Out}(i)$. If channel e is an incoming(outgoing) channel of node i , we denote it by $\text{Head}(e) = i$ ($\text{Tail}(e) = i$), respectively. We assume the discussion in this paper is restricted to a single-source acyclic network. Let $\text{In}(S)$ denote a set of *imaginary channels*, which terminate at the source node S and the number of these imaginary channels is context dependent and always denoted by ω . A channel may fail due to various reasons in a real network. In [7], the status of the network can be specified by a mapping $\varepsilon: E \rightarrow \{0, 1\}$ called a *configuration*. Channels being in set $\varepsilon^{-1}(0)$ are failed channels with respect to the configuration ε . The ε -*subnetwork* is the network resulting from the elimination of failed channels. For a ε -*subnetwork*, we denote max-flow of a non-source node t by $\text{maxflow}_\varepsilon(t)$. Similarly, the max-flow of a collection T of non-source nodes is denoted by $\text{maxflow}_\varepsilon(T)$ and the max-flow of a collection $\wp$ of channels is denoted by $\text{maxflow}_\varepsilon(\wp)$. The total number of configurations is $2^{|E|}$, where $|E|$ is the number of all channels. Let F be a finite field and ε be a configuration. A scalar $k_{d,e} \in F$ for every adjacent pair (d, e) is called the *local encoding kernel* and the ε -*global encoding kernels* of channel e , denoted by $f_{e,\varepsilon}$, can be calculated recursively in an upstream-to-downstream order by

$$f_{e,\varepsilon} = \varepsilon(e) \sum_{d \in \text{In}(t)} k_{d,e} f_{d,\varepsilon} \text{ for } e \in \text{Out}(t).$$

Particularly, a channel e with $\varepsilon^{-1}(0)$ has $f_{e,\varepsilon} = 0$ and hence transmits symbol $x \cdot f_{e,\varepsilon} = 0$.

A sequence of channels $\{e_1, e_2, \dots, e_n\}$ leading to a node T_q where each $e_i \in \text{In}(T_i) \cap \varepsilon^{-1}(1)$, is called a *path* such that $e_1 \in \text{In}(S)$, $e_n \in \text{In}(T_q)$, and $\text{Head}(e_i) = \text{Tail}(e_{i+1})$ for all i . If two paths do not share any channel, they are *channel-disjoint*.

paths or are said to be *channel-disjoint*. An arbitrary collection of channels $E = \{e_1, e_2, \dots, e_n\}$, where each $e_i \in E \cap \varepsilon^{-1}(1)$ for all $1 \leq i \leq n$, qualifies as an *independent collection* if each $e \in E$ is on a path and these paths are channel-disjoint. The channel-disjoint paths for an independent collection are called an *associated flow* [6]. An independent collection may have more than one associated flow. An independent collection is a *regular* independent collection or is said to be *regular* if the corresponding ε -global encoding kernels are linearly independent. Let $\alpha = \{e_1, e_2, \dots, e_m\}$ be an independent collection, where $m \geq 1$, and $\beta = \{e_1^p, e_2^p, \dots, e_n^p\}$ be an independent collection, where $n \geq 1$. If a channel $e \in \alpha$ and a channel $e^p \in \beta$ form an adjacent pair (e, e^p) , namely $\text{Head}(e) = \text{Tail}(e^p)$, then independent collection α is said to support independent collection β , denoted by $\alpha \rightarrow \beta$.

III. PROBLEM FORMULATION

The formal definitions of static network codes were first defined by Yeung *et al.* [7]. We refer the reader to [7] (see also [8]) for these definitions. It is easy to see that original definitions seem to only concern the vector space spanned by incoming ε -global encoding kernels associated with certain collections of nodes rather than the independence structure of these codes, which is referred to as the *node-based approach* in [6]. The node-based approach has a limitation on describing static network codes from an integrated perspective, which can be illustrated by Figure 1.

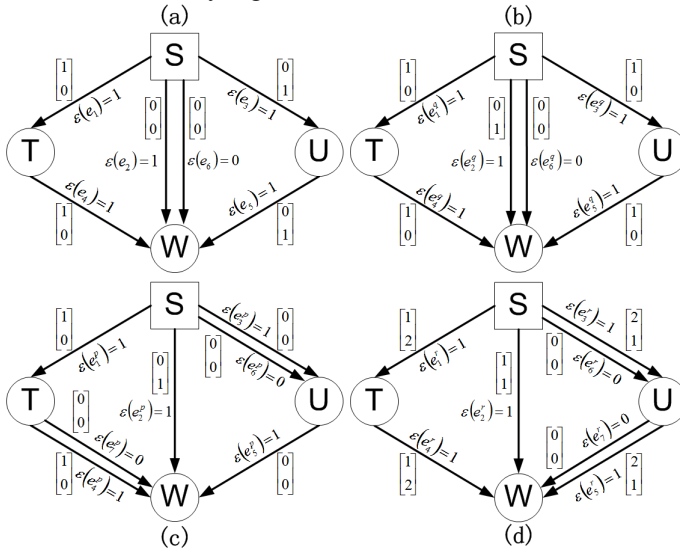


Figure 1. (a) is a 2-dimensional binary static linear dispersion, (b) is a 2-dimensional binary static linear broadcast, (c) is a 2-dimensional binary static linear multicast, and (d) is a 2-dimensional ternary static generic linear network code.

We can verify from Figure 1 that, for every configuration ε , according to the original definitions of static network codes [7], every node W has $\dim(V_{W,\varepsilon}) = 2$ even (a), (b), (c) and (d) are actually different class of static network codes. The

inherent limitation of node-based approach makes us fail to capture the independence structure of these codes. In other words, we are unable to distinguish between these static network codes from the original definitions.

Originally, the static generic network code is proposed with aim to facilitate the construction of the rest. Indeed, it requires every collection of ε -global encoding kernels that can possibly be linearly independent are linearly independent, thus the existence of a static generic network code is able to ensure the existence of the rest. However, under what condition the ε -global encoding kernels that can possibly be linearly independent is not explicitly elaborated in [7], which makes the definition arduous to be understood. Besides, the definition does not provide an effective and intuitive approach to verify such a code (Figure 1 is such an example). In the rest of this section, we are dedicated to removing the ambiguity.

We first establish a specific form of the max-flow bound in following theorem, which can be applied to static network coding (Following [8, Theorem 19.10]).

Theorem 1: Let ε be a configuration of an acyclic network. For an ω -dimensional linear network code on the network, for any collection T of non-source nodes,

$$\dim(V_{T,\varepsilon}) \leq \min\{\omega, \max\text{flow}_{\varepsilon}(T)\}.$$

Proof: Let the acyclic network be $G = (V, E)$ and the configuration be ε . Let ζ be a cut between source node S and a collection T of non-source nodes, and E_{ζ} be the collection of channels across the cut ζ . Then $V_{T,\varepsilon}$ is a linear transformation of $V_{E_{\zeta},\varepsilon}$, where

$$\dim(V_{T,\varepsilon}) \leq \dim(V_{E_{\zeta},\varepsilon}) \leq |E_{\zeta}|.$$

We minimize over all the cuts between S and T and denote the minimal cut by $\text{Mincut}_{\varepsilon}(S, T)$. From the Max-Flow Min-Cut theorem, we know that the information rate received by a collection T obviously cannot exceed the minimal cut, and then we have

$$\dim(V_{T,\varepsilon}) \leq \max\text{flow}_{\varepsilon}(T) = \text{Mincut}_{\varepsilon}(S, T).$$

On the other hand, $V_{T,\varepsilon}$ is a linear transformation of $V_S = F^{\omega}$. Consequently, $\dim(V_{T,\varepsilon}) \leq \dim(V_S) = \omega$. Theorem is proved $\square$

We can imply from above theorem that, for a collection $\wp$ of channels other than the imaginary channels, we have $\dim(V_{\wp,\varepsilon}) \leq \min\{\omega, \max\text{flow}_{\varepsilon}(\wp)\}$. A direct consequence of this fact is the lemma 1 below, which states a necessary condition for a collection of ε -global encoding kernels to be linearly independent.

Lemma 1: Let ε be a configuration of an acyclic network. If an arbitrary collection of channels $E = \{e_1, e_2, \dots, e_m\}$, where each $e_i \in E \cap \varepsilon^{-1}(1)$ for all $1 \leq i \leq m$, such that their corresponding ε -global encoding kernels $f_{e_1,\varepsilon}, f_{e_2,\varepsilon}, \dots, f_{e_m,\varepsilon}$ are linearly independent, then the collection E is an independent collection.

Proof: Let ε be a configuration and an arbitrary collection of channels be $E = \{e_1, e_2, \dots, e_m\}$ with their corresponding ε -global encoding kernels being linearly independent. We install a new node t to the network connecting $\text{Head}(e_i)$ to node t by a new channel e_i^p and letting $f_{e_i, \varepsilon} = f_{e_i^p, \varepsilon}$ for all $1 \leq i \leq m$ respectively. Let ζ be a cut between S and t and E_ζ be the collection of channels across the cut ζ . Then $V_{t, \varepsilon}$ is a linear transformation of V_{E_ζ} , where

$$\dim(V_{t, \varepsilon}) \leq \dim(V_{E_\zeta}) \leq |E_\zeta|.$$

We minimize over all the cuts between S and t and denote the minimal cut by $\text{Mincut}_\varepsilon(S, t)$. From the Max-Flow Min-Cut theorem, we know that the information rate received by a node t obviously cannot exceed the minimal cut, and then we have $\dim(V_{t, \varepsilon}) \leq \text{Mincut}_\varepsilon(S, t)$. Let ζ^p be the cut between S and t such that $E_{\zeta^p} = \{e_i^p : 1 \leq i \leq m\}$, thus we have

$$\dim(V_{t, \varepsilon}) = m \leq \text{Mincut}_\varepsilon(S, t) \leq |E_{\zeta^p}| = m.$$

Invoking the Max-Flow Min-Cut theorem, we can obtain $\text{maxflow}_\varepsilon(t) = \text{Mincut}_\varepsilon(S, t) = m$. Thus there exists a collection of channel-disjoint paths from the imaginary channels to node t . Conceivably, if there exists a path from S to t that goes through the channel e_i^p must also go through the channel e_i . Therefore, we can come to the conclusion that there must exist m channel-disjoint paths starting from m of the imaginary channels and terminating at the channels in the collection of E , and E is an independent collection. $\square$

The above lemma actually establishes a strong relationship between the linear independence among ε -global encoding kernels and graph theory which implies that whenever an arbitrary collection of ε -global encoding kernels can possibly be linearly independent only on condition that their corresponding channels can construct an independent collection. This lemma gives rise to the possibility of constructing a best static linear network codes regarding linear independence is to form a regular independent collection, which is stated by below lemma. We prove it by using the algorithm in [10].

Lemma 2: Let φ be any arbitrary nonempty set of independent collections and q be the size of a base finite field F . There exists a static network on an acyclic network with N channels such that any independent collection in φ is regular if $q \geq 2^{|N|}|\varphi|$.

Proof: By definition, the ε -global encoding kernels for the imaginary channels form the standard basis and each independent collection in φ has an associated flow. We construct the ε -global encoding kernels by an upstream-to-downstream order as the algorithm in [10]. For each

associated flow, a *border collection*, constructed by the last processed channels on its paths, is an independent collection. The goal of our algorithm is to show the preservation of each border collection as a regular independent collection. Obviously, each border collection, at the beginning, associated with each independent collect in φ is a subset of all the imaginary channels. Thus each border collection is a regular independent collection to begin with. Assume that the regularity of all the border collections is maintained at the current step. Let e be the next channel to be processed and n be the number of new border collections β_i given rise to by channel e . We assume that α_i supports β_i , namely $\alpha_i \rightarrow \beta_i$ for $1 \leq i \leq n$ and the α_i , where $1 \leq i \leq n$, is the border collection in the current step. We denote the only channel that belongs to α_i but does not belong to β_i by $e_i = \alpha_i \setminus \beta_i$ and $\text{Tail}(e) = t$. Given that for all $1 \leq i \leq n$, $f_{e_i, \varepsilon} \in V_{t, \varepsilon}$ and the ε -global encoding kernels of e_i and $\alpha_i \setminus \beta_i$ are both linearly independent by the induction assumption, we obtain $V_{t, \varepsilon} \setminus \{f_{e, \varepsilon} : e \in \alpha_i, e \notin \beta_i\}$ is nonempty. This implies that $\dim(V_{t, \varepsilon} \cap \{f_{e, \varepsilon} : e \in \alpha_i, e \notin \beta_i, 1 \leq i \leq n\}) \leq \dim(V_{t, \varepsilon}) - 1$. If

the base field $q = |F| \geq 2^{|N|}|\varphi|$, where $|N|$ is the number of channels in the network. Then we obtain

$$\begin{aligned} \sum_{\varepsilon} |V_{t, \varepsilon} \cap [\bigcup_{1 \leq i \leq n} \{f_{e, \varepsilon} : e \in \alpha_i, e \notin \beta_i\}]| \\ \leq \sum_{\varepsilon} n \times q^{\dim(V_{t, \varepsilon}) - 1} \\ \leq 2^N n \times q^{\dim(V_{t, \varepsilon}) - 1}. \end{aligned}$$

From the above we can see if the base field size satisfies the condition that $q = |F| \geq 2^N|\varphi|$, we can always choose a vector in $V_{t, \varepsilon} \setminus \bigcup_{\varepsilon} (V_{t, \varepsilon} \cap \{f_{e, \varepsilon} : e \in \alpha_i, e \notin \beta_i, 1 \leq i \leq n\})$ to be the ε -global encoding kernel of channel e , thus the regularity of the new border collections can be maintained. As a consequence, all the independent collections in φ are regular as the algorithm is finished. $\square$

By far, we have established a strong relationship that exists between static network codes and graph theory by lemma 1 and lemma 2. Conceivably, there exists a class of static linear network code with every independent collection being regular. In the following, we will prove that such a code is just a static generic linear network code and a static generic linear network code is obliged to make every independent collection regular. We provide some new alternative definitions of static generic network code below by using the independent collection and prove them equivalent to the original one. We assume that there are no less than $|N(S)|$ outgoing channels from source node S being in $\varepsilon^{-1}(1)$, by which all information generated by source node S can be properly propagated over the network.

Definition 1: Let ε be a configuration. If an independent collection I has ω channels in $\varepsilon^{-1}(1)$, then its corresponding ε -global encoding kernels $\{f_{e,\varepsilon} : e \in I\}$ are linearly independent.

Definition 2: Let ε be a configuration. If an independent collection E has m channels in $\varepsilon^{-1}(1)$, where $m \leq \omega$, then its corresponding ε -global encoding kernels $\{f_{e,\varepsilon} : e \in E\}$ are linearly independent.

Definition 3: Let ε be a configuration and $\{e_1, e_2, \dots, e_m\}$ be a collection of channels with their corresponding ε -global encoding kernels being $f_{e_1,\varepsilon}, f_{e_2,\varepsilon}, \dots, f_{e_m,\varepsilon}$, where $m \leq \omega$. If $f_{e_1,\varepsilon}, f_{e_2,\varepsilon}, \dots, f_{e_{m-1},\varepsilon}$ are linearly independent, $V_{T_m,\varepsilon} \not\subset \langle f_{e_1,\varepsilon}, f_{e_2,\varepsilon}, \dots, f_{e_{m-1},\varepsilon} \rangle$ and there does not exist a directed path from T_m to T_j for all $1 \leq j \leq m-1$, where each $e_i \in \text{Out}(T_i) \cap \varepsilon^{-1}(1)$ for all $1 \leq i \leq m$, then $f_{e_1,\varepsilon}, f_{e_2,\varepsilon}, \dots, f_{e_m,\varepsilon}$ are linearly independent.

Definition 4: Let ε be a configuration and $\{e_1, e_2, \dots, e_m\}$ be a collection of channels with their corresponding ε -global encoding kernels being $f_{e_1,\varepsilon}, f_{e_2,\varepsilon}, \dots, f_{e_m,\varepsilon}$, where $m \leq \omega$. If $V_{T_m,\varepsilon} \not\subset \langle f_{e_1,\varepsilon}, f_{e_2,\varepsilon}, \dots, f_{e_{m-1},\varepsilon} \rangle$ and there does not exist a directed path from T_m to T_j for all $1 \leq j \leq m-1$, where each $e_i \in \text{Out}(T_i) \cap \varepsilon^{-1}(1)$ for all $1 \leq i \leq m$, then $f_{e_1,\varepsilon}, f_{e_2,\varepsilon}, \dots, f_{e_m,\varepsilon}$ are linearly independent.

Definition 5: Let ε be a configuration and $\{e_1, e_2, \dots, e_m\}$ be a collection of channels, where each $e_i \in \text{Out}(T_i) \cap \varepsilon^{-1}(1)$ for all $1 \leq i \leq m$. Then, the corresponding ε -global encoding kernels $f_{e_1,\varepsilon}, f_{e_2,\varepsilon}, \dots, f_{e_m,\varepsilon}$ are linearly independent ($m \leq \omega$) provided that $V_{T_j,\varepsilon} \not\subset \langle f_{e_k,\varepsilon} : k \neq j \rangle$ for all j .

Remark: Definition 5 is the original definition of static generic network code in [7]. Note that it is necessary to impose the constraint that there does not exist a directed path from T_m to T_j for all $1 \leq j \leq m-1$ in definitions (3) and (4). Otherwise, it is impossible to hold the equivalence between those conditions.

Proof: We will prove that (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3), (3) $\Rightarrow$ (4), (4) $\Rightarrow$ (5) and (5) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2): If there exists an independent collection E which has m channels, where $m < \omega$, then we can always add $\omega - m$ channels deriving from the imaginary channel to this independent collection until it contains ω channels. Now that (1) holds and E is a subset of I , it is easy to obtain that corresponding ε -global encoding kernels $\{f_{e,\varepsilon} : e \in E\}$ are also linearly independent. Then we proved (1) implies (2).

(2) $\Rightarrow$ (3): Let ε be a configuration and $\{e_1, e_2, \dots, e_m\}$ be a collection of channels with their corresponding ε -global encoding kernels being $f_{e_1,\varepsilon}, f_{e_2,\varepsilon}, \dots, f_{e_m,\varepsilon}$, where $m \leq \omega$, such that $f_{e_1,\varepsilon}, f_{e_2,\varepsilon}, \dots, f_{e_{m-1},\varepsilon}$ are linearly independent, $V_{T_m,\varepsilon} \not\subset \langle f_{e_1,\varepsilon}, f_{e_2,\varepsilon}, \dots, f_{e_{m-1},\varepsilon} \rangle$ and there does not exist a directed path from T_m to T_j for all $1 \leq j \leq m-1$, where each $e_i \in \text{Out}(T_i) \cap \varepsilon^{-1}(1)$ for all $1 \leq i \leq m$. Given that $V_{T_m,\varepsilon} \not\subset \langle f_{e_1,\varepsilon}, f_{e_2,\varepsilon}, \dots, f_{e_{m-1},\varepsilon} \rangle$, we can always find a channel $e_m^p \in \text{In}(T_m)$ such that $f_{e_1,\varepsilon}, f_{e_2,\varepsilon}, \dots, f_{e_{m-1},\varepsilon}, f_{e_m^p,\varepsilon}$ are linearly independent. According to Lemma 1, each channel in $\{e_1, e_2, \dots, e_m^p\}$ is on a channel-disjoint path from imaginary channels, namely $\{e_1, e_2, \dots, e_m^p\}$ is an independent collection. Because there does not exist a directed path from T_m to T_j for all $1 \leq j \leq m-1$, the collection $\{e_1, e_2, \dots, e_m\}$ must also be an independent collection. If (2) holds, $f_{e_1,\varepsilon}, f_{e_2,\varepsilon}, \dots, f_{e_m,\varepsilon}$ are linearly independent. Then we proved (2) implies (3).

(3) $\Rightarrow$ (4): We first assume that (3) holds and defines a static linear code. Suppose that $\alpha = \{e_1, e_2, \dots, e_{m-1}\}$ is any collection of channels with their corresponding ε -global encoding kernels not being necessarily linearly independent. Let any channel $e_m \notin \alpha$ such that $V_{T_m,\varepsilon} \not\subset \langle f_{e_1,\varepsilon}, f_{e_2,\varepsilon}, \dots, f_{e_{m-1},\varepsilon} \rangle$. Then we can always find a subset $\alpha^p \subset \alpha$ such that $\dim(V_{\alpha^p,\varepsilon}) = \dim(V_{\alpha,\varepsilon})$ and $\{f_{e^p,\varepsilon} : e^p \in \alpha^p\}$ are linearly independent. If (3) holds, we have $f_{e_m,\varepsilon} \notin V_{\alpha^p,\varepsilon}$ and $f_{e_m,\varepsilon} \notin V_{\alpha,\varepsilon}$. Then this linear code also satisfies (4). We conclude that (3) implies (4).

(4) $\Rightarrow$ (5): We prove this by two steps. In the first step, we will prove by contradiction that there exists at least one k such that there is no directed path from T_k to T_j for all $j \neq k$, where $j \geq 1$ and $m \geq k$. We first assume that there is at least one directed path from T_k to T_j for all k and some $j \neq k$. On condition that the collection $\{T_k : 1 \leq k \leq m\}$ is a finite collection, whichever node T_k we choose to start to go through such directed paths, we definitely go back to the node T_k where we start, which means there is a directed cycle in the network, a contradiction to the fact that the network is acyclic. Therefore, the existence of k is proved.

In the second step, we prove this by mathematical induction on m . We first assume the assertion is true for $m-1$ for $2 \leq m \leq \omega$, and then we need to prove the assertion is also true for m . Let ε be a configuration. We suppose

$\alpha = \{e_1, e_2, \dots, e_m\}$ is a collection of channels with corresponding ε -global encoding kernels being $f_{e_1, \varepsilon}, f_{e_2, \varepsilon}, \dots, f_{e_m, \varepsilon}$ and $V_{T_j, \varepsilon} \not\subset V_{\alpha_j, \varepsilon}$ for all $1 \leq j \leq m$. Applying the proved k in the first step, we denote $\alpha_k = \{e_i \in \alpha : i \neq k\}$. Then we have $f_{e_k, \varepsilon} \notin V_{\alpha_k, \varepsilon}$ and $V_{\alpha_k, \varepsilon} = \langle \{f_{e_i, \varepsilon} : e_i \in \alpha_k\} \rangle$. For the reason that $V_{T_j, \varepsilon} \not\subset V_{\alpha_j, \varepsilon}$ and $V_{\alpha_k \setminus \{e_j\}, \varepsilon} = V_{\alpha_j \setminus \{e_k\}, \varepsilon} \subset V_{\alpha_j, \varepsilon}$, we have $V_{T_j, \varepsilon} \not\subset V_{\alpha_k \setminus \{e_j\}, \varepsilon}$ for all $j \neq k$. We apply the induction hypothesis to α_k , and obtain that $\{f_{e, \varepsilon} : e \in \alpha_k\}$ are linearly independent, thus $f_{e_1, \varepsilon}, f_{e_2, \varepsilon}, \dots, f_{e_m, \varepsilon}$ are linearly independent. (4) implies (5).

(5) $\Rightarrow$ (1): We omit this proof here since it has been previously proved in [9].

We have proved that the five definitions are equivalence. These definitions can meet the needs for various requirements. In particular, the definition 1 suggests another way of constructing a static generic network code, on which we only need to concern the independent collections with ω channels

and by Lemma 2, the required field size is at least $2^N \binom{N}{\omega}$,

where N is the number of channels in $\varepsilon^{-1}(1)$. The definition 1 also provides a straightforward way to verify such a code. The definition 2 is more intuitional and easy to understand, and the definition 4 is more convenient for constructing a static generic linear network code, which implies when to construct such a code, a new added channel should carry linearly independent ε -global encoding kernels base on the existence whenever possible.

IV. AN INTEGRATED SCHEME

In previous section, we have obtained the necessary and sufficient conditions for a collection of ε -global encoding kernels to be possibly linearly independent. Based on the results obtained, it is possible that all different classes of static network codes can be described and constructed under an integrated scheme. In this section, we will propose such an integrated scheme for static network codes. As we will see later, by original definitions of static network codes, we cannot describe and construct these codes in an integrated approach, but with our scheme we can do so. The introduced tool of independent collection gives us a better position that affords an extensive overall view of static network codes from linear independence perspective. All the information regarding linear independence among ε -global encoding kernels can be covered by this scheme.

Recall the example in Figure 1, by original definition of static network code [7], we are unable to distinguish between these static network codes; however, if we introduce the tool of independence collection to describe these static network codes, we find that the regular independent collections corresponding to the static network code in Figure 1 (a) are

$$\{e_2\}, \{e_4\}, \{e_5\}, \{e_4, e_5\},$$

the regular independent collections corresponding to the static network code in Figure 1 (b) are

$$\{e_2^q\}, \{e_4^q\}, \{e_5^q\}, \{e_2^q, e_4^q\}, \{e_2^q, e_5^q\},$$

the regular independent collections corresponding to the static network code in Figure 1 (c) are

$$\{e_2^p\}, \{e_4^p\}, \{e_5^p\}, \{e_2^p, e_4^p\},$$

and the regular independent collections corresponding to the static network code in Figure 1 (d) are

$$\{e_2^r\}, \{e_4^r\}, \{e_5^r\}, \{e_2^r, e_4^r\}, \{e_2^r, e_5^r\}, \{e_4^r, e_5^r\}.$$

We know by intuition that a static generic linear network code requires the most strict linear independence among the ε -global encoding kernels, thus it must have the largest number of regular independent collections when described by regular independent collections. The above results obtained correspond to our intuition. Hence, in contrast with the node-based approach which does not distinguish these static network codes, we are now able to distinguish them by the regular independent collections.

We have seen that a static generic linear network code is described by regular independent collections in previous section. Similarly, in the rest of this section, we will prove that a static linear dispersion, a static linear broadcast and a static linear multicast can also be described by regular independent collections. The theorem below provides an alternative definition of a static linear dispersion in terms of regular independent collections, which has been proved to be equivalent to its original definition.

Theorem 2: A ω -dimensional F -valued linear network code qualifies as a static linear dispersion *if and only if* for every configuration ε and every collection $\wp$ of non-source nodes, there exists a regular independent collection $\ell_\wp$ with at least $\min\{\omega, \maxflow_\varepsilon(\wp)\}$ channels in $\varepsilon^{-1}(1)$ such that $\text{Head}(e) \in \wp$ for all $e \in \ell_\wp$.

Proof: $\dim(\langle \bigcup_{T \in \wp} V_{T, \varepsilon} \rangle) = \min\{\omega, \maxflow_\varepsilon(\wp)\}$ implies that any subsets $\ell_\wp$ of $\bigcup_{T \in \wp} \text{In}(t)$ channels that satisfy $|\ell_\wp| = \min\{\omega, \maxflow_\varepsilon(\wp)\}$ are with their corresponding ε -global encoding kernels $\{f_{e, \varepsilon} : e \in \ell_\wp\}$ being linearly independent. Consequently, $\ell_\wp$ must be a regular independent collection. In reverse, if there exists a regular independent collection $\ell_\wp$ with $\min\{\omega, \maxflow_\varepsilon(\wp)\}$ channels such that $\text{Head}(e) \in \wp$ for all $e \in \ell_\wp$, it actually implies

$$\dim(\bigcup_{T \in \wp} V_{T, \varepsilon}) \geq \min\{\omega, \maxflow_\varepsilon(\wp)\}.$$

Using the technique to prove lemma 1 and theorem 1, we can obtain

$$\dim\left(\bigcup_{T \in \emptyset} V_{T, \varepsilon}\right) \leq \min\{\omega, \max\text{flow}_{\varepsilon}(\emptyset)\}.$$

Based upon the two inequations, we can conclude that

$$\dim\left(\bigcup_{T \in \emptyset} V_{T, \varepsilon}\right) = \min\{\omega, \max\text{flow}_{\varepsilon}(\emptyset)\},$$

which is original definition of a static linear dispersion. The theorem is proved. $\square$

The above theorem actually associates a static linear dispersion with regular independent collections and describes a static linear dispersion by the regular independent collections.

In a similar fashion, we can provide alternative definitions of a static linear broadcast and a static linear multicast below.

Theorem 3: A ω -dimensional F -valued linear network code qualifies as a static linear broadcast *if and only if* for every configuration ε and every non-source node T , there exists a regular independent set ℓ_T with at least $\min\{\omega, \max\text{flow}_{\varepsilon}(T)\}$ channels in $\varepsilon^{-1}(1)$ such that $\text{Head}(e) = T$ for all $e \in \ell_T$.

Theorem 4: A ω -dimensional F -valued linear network code qualifies as a static linear multicast *if and only if* for every configuration ε and every non-source node T with at least $\max\text{flow}_{\varepsilon}(T) \geq \omega$, there exists a regular independent set ℓ_T with ω channels in $\varepsilon^{-1}(1)$ such that $\text{Head}(e) = T$ for all $e \in \ell_T$.

Since Theorem 3 and Theorem 4 is the straightforward corollary of Theorem 2, we omit the correspond proofs.

By far, we have proved that all different classes of static network codes can be described by an integrated scheme in terms of regular independent collections.

Consequently, we have obtained an integrated scheme for static network codes under which all different classes of static network codes can be described and constructed. Below, we make use of the advantage of the integrated scheme to simplify some proofs. We will prove the fact that *a static generic linear network code is a static linear dispersion*. Without the integrated scheme, the proofs will be pretty complicated and not intuitional. A simpler proof based on the integrated scheme is provided below.

Theorem 5: A static generic linear network code is a static linear dispersion.

Proof: The Lemma 2 tells us that a static generic linear network code makes any independent collections in $\emptyset$ regular, which abides with the Theorem 2, the definition of a static linear dispersion network code, so this static generic linear network code is also a static linear dispersion. $\square$

V. CONCLUSIONS

In this paper, we first analyzed the limitations of the original definition of static network codes. Aiming at these limitations, we established a strong relationship that exists

between static network codes and graph theory and obtained the necessary and sufficient conditions for a collection of ε -global encoding kernels to be possibly linearly independent. Base on that, we introduced some new alternative definitions of a static generic network code and proved them equivalent to the original one. Afterwards corresponding substituted definitions for other static network codes were given. Finally, an integrated scheme for static network codes was proposed, under which all different classes of static network codes can be described, constructed and verified in an integrated way. To find an application of our results, we made use of the advantage of the integrated scheme to simplify some proofs that used to be complicated.

In practical situation, a practical application may deal with a certain collection $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k\}$ of configurations [8], where

$k \ll 2^{|N|}$ and $|N|$ is the total number of channels in the network, in order to provide link contingency, network security, network expandability, transmission redundancy, etc. Thus it is plausible that we may define an $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k\}$ -static linear codes with less requirement on the sufficient size of the base field as well as the low computational complexity. Conceivably, we can apply the results obtained in this paper to an $\{\varepsilon_1, \varepsilon_2, \dots, \varepsilon_k\}$ -static linear codes.

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