

Mathematical Model for Bit-error-rate of a Spread-Spectrum System

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Abstract - Bit-error-rate (BER) is known to be an important index for measuring the performance of wireless communication systems. The use of software simulation is common for modelling BER characteristics of spread-spectrum systems, but analytic solutions for the same are not readily available. This paper presents a mathematical (analytic) model developed by the author for BER of a spread-spectrum system. The model was tested for different sets of even- and odd-degree Gold codes, generated from appropriate combination of preferred pairs of m-sequences. The outcome of the mathematical model was compared with results of software simulation carried out for the system performance. The two sets of results were found to agree closely, thereby validating the mathematical model.

Keywords - Spread-spectrum system; analytic model; cross-correlation index; bit-error-rate, Gold codes.

I. INTRODUCTION

Spread spectrum is a unique technique which was developed originally for military applications but later became significant worldwide communication technique in the larger society for commercial and civilian applications. It is an important technique in wireless telephony as well as satellite navigation. Although the popularity of spread spectrum technique in mobile telephony has been diminishing in recent times, it remains a significant modality for radio communication in the global positioning system (GPS). It is also used in other areas of application [1-3] including tomography imaging [4], digital terrestrial television [5], anti-jam underwater communication [6] and in encryption of medical images [7].

Around the turn of the twenty-first century, another technique called interleave-division multiple-access (IDMA) was proposed in literature [1-4] as an alternative to spread-spectrum technology. IDMA avoids spectrum-spreading thereby minimizing transmission bandwidth and computational requirements [4-7]. These are positive advantages of IDMA, but they come with significant trade-offs, including loss of jam-resistant properties and capability for covert transmission possessed by spread spectrum techniques.

In recent times, advanced hybrids of spread-spectrum systems have been realized, involving combination of spread-

spectrum communication with other channel enhancement techniques like multicarrier transmission (or orthogonal frequency-division multiplexing – OFDM), multiple-input multiple-output (MIMO) antenna system; different variants of space-time coding, including space-time block codes, space-time trellis codes and super-orthogonal space-time trellis codes [8-12]. However, these are not the focus of this paper.

Bit-error-rate (BER) is a percentage measure of number of bits in error that occurred during transmission of information from a transmitter to a receiver. It is an important index for measuring the performance of a wireless communication system. For spread-spectrum systems, properties of spreading codes (or sequences) are important factors affecting the system BER. To determine BER of such systems, software simulations are often used, as analytic solutions are not readily available.

This paper presents a mathematical (analytic) model for BER and multiple-access performance of a spread-spectrum system. This paper also presents results of tests carried out by the author for the model for different sets of even- and odd-degree Gold codes for the system. The model is compared with results of simulation carried out for the system performance.

This work is part of a larger study. The rest of this paper is organized as follows. Starting from basic principles, a mathematical model is derived for the system. This is followed by a presentation of validation method adopted for this work. Results of the analysis are then presented, followed by a brief conclusion.

II. SYSTEM MODEL

Consider a direct-sequence spread-spectrum (DSSS) system. Spread spectrum signal transmitted by a user k can be expressed as

$$s_k(t) = \frac{A}{N} c_k(t) b_k(t) \cos(\omega_c t + \theta_k), \quad (1)$$

where $b_k(t)$ is the user binary data, $c_k(t)$ is the spreading code, ω_c is the carrier frequency and N is length of the code. Here, N serves as a normalization factor to keep bit-energy constant

regardless of the length of the spreading sequence. The spreading code $c_k(t)$ for the user can be denoted as

$$c_k(t) = \sum_{i=1}^N c_k^i P_c(t - iT_c), \quad c_k^i \in \{-1, +1\}, \quad (2)$$

where P_c is a rectangular pulse having a duration T_c . Let the wireless communication channel be represented by multiple paths having a real positive gain β_l , propagation delay τ_l and phase shift γ_l , where l is the path index. The channel impulse response $h_k(t)$ for L independent paths can be modelled as

$$h_k(t) = \sum_{l=1}^L \beta_{kl} e^{j\gamma_{kl}} \delta(t - \tau_{kl}) \quad (3)$$

At the receiving end, the received signal $r_k(t)$ for the user is obtained by convolving $s_k(t)$ with $h_k(t)$:

$$r_k(t) = \frac{A}{N} \sum_{l=1}^L \beta_{kl} e^{j\gamma_{kl}} c_k(t - \tau_{kl}) b_k(t - \tau_{kl}) \cos(\omega_c t - \theta_{kl}) \quad (5)$$

$$r(t) = \frac{A}{N} \sum_{k=1}^K \sum_{l=1}^L \beta_{kl} e^{j\gamma_{kl}} c_k(t - \tau_{kl}) b_k(t - \tau_{kl}) \cos(\omega_c t - \theta_{kl}) \quad (6)$$

$$z_1(m) = \int_{mT_b}^{(m+1)T_b} (r(t) + n(t)) c_1(t - \tau_{k1}) \cos(\omega_c t - \theta_{k1}) dt \quad (7)$$

$$z_1(m) = \frac{A}{N} \int_{mT_b}^{(m+1)T_b} \left\{ \sum_{k=1}^K \sum_{l=1}^L \beta_{kl} e^{j\gamma_{kl}} c_k(t - \tau_{kl}) b_k(t - \tau_{kl}) \cos(\omega_c t - \theta_{kl}) \right\} c_1(t - \tau_{k1}) \cos(\omega_c t - \theta_{k1}) dt \\ + \int_{mT_b}^{(m+1)T_b} n(t) c_1(t - \tau_{k1}) \cos(\omega_c t - \theta_{k1}) dt \quad (8)$$

$$z_1(m) = \frac{A}{N} \int_{mT_b}^{(m+1)T_b} \left\{ \sum_{l=1}^L \beta_{1l} e^{j\gamma_{1l}} c_1(t - \tau_{1l}) b_1(t - \tau_{1l}) \cos(\omega_c t - \theta_{1l}) \right\} c_1(t - \tau_{1l}) \cos(\omega_c t - \theta_{1l}) dt \\ + \frac{A}{N} \int_{mT_b}^{(m+1)T_b} \left\{ \sum_{k=2}^K \sum_{l=1}^L \beta_{kl} e^{j\gamma_{kl}} c_k(t - \tau_{kl}) b_k(t - \tau_{kl}) \cos(\omega_c t - \theta_{kl}) \right\} c_1(t - \tau_{1l}) \cos(\omega_c t - \theta_{kl}) dt \\ + \int_{mT_b}^{(m+1)T_b} n(t) c_1(t - \tau_{k1}) \cos(\omega_c t - \theta_{k1}) dt \\ = z_{11} + z_{12} + z_{13} \quad (9)$$

Substituting the expressions for $s_k(t)$ and $h_k(t)$ into this integral, and using relevant properties of the Dirac delta function $\delta(t)$ gives (5). For a multi-user system comprising K users, the received signal $r(t)$ is a linear superposition of the signals for the users, and is given by (6).

Let user-1 be the reference user. Assuming coherent demodulation, the receiver output $z(m)$ for m^{th} bit during the bit duration T_b of the user is given by (7), where $n(t)$ is receiver noise. Substituting for $r(t)$ in (7) gives (8). Define $k = 1$ for the reference user. Using this in (8) gives (9), where z_{11} represents the desired signal for the reference user, z_{12} is interference term, and z_{13} is noise term.

$$\begin{aligned}
z_{11} &= \frac{A}{N} \int_{mT_b}^{(m+1)T_b} \left\{ \sum_{l=1}^L \beta_{1l} e^{j\gamma_{1l}} c_1(t - \tau_{1l}) b_1(t - \tau_{1l}) \cos(\omega_c t - \theta_{1l}) \right\} c_1(t - \tau_{1l}) \cos(\omega_c t - \theta_{1l}) dt \\
&= \frac{A}{2N} \sum_{l=1}^L \beta_{1l} e^{j\gamma_{1l}} \int_{mT_b}^{(m+1)T_b} c_1^2(t - \tau_{1l}) b_1(t - \tau_{1l}) [1 + \cos 2(\omega_c t - \theta_{1l})] dt
\end{aligned} \quad (10)$$

Consider z_{11} (10). Using the fact that the term $c_1^2(t - \tau_{1l})$ represents the inner product of the first user-assigned code $c_1(t)$ with itself, and choosing ω_c such that $f_c = \frac{n}{T_b}$ ($n \in \mathbf{Z}$), we can show that z_{11} reduces to:

$$z_{11} = \frac{A}{2} \sum_{l=1}^L \beta_{1l} e^{j\gamma_{1l}} \left\{ \int_{mT_b}^{(m+1)T_b} b_1(t - \tau_{1l}) dt \right\}. \quad (11)$$

Assume that $b_1(t - \tau_{1l}) = 1$ in the period T_b . For a Gaussian channel, number of paths $L = 1$. Consequently, there is no multipath fading. Therefore, $\beta_{1l} = 1$ and $\gamma_{1l} = 0$. With these, we can show that for a Gaussian channel, the last equation reduces to:

$$z_{11} = \frac{AT_b}{2}. \quad (12)$$

Similarly, we can show that the interference term z_{12} will reduce to:

$$z_{12} = \pm \frac{A}{2N} \sum_{k=2}^K \sum_{l=1}^L \beta_{kl} e^{j\gamma_{kl}} \left\{ \int_{mT_b}^{(m+1)T_b} d_{1k} dt \right\}, \quad (13)$$

where $d_{1k} = c_1(t - \tau_{1l})c_k(t - \tau_{kl})$, the cross-correlation between the code sequence for user-1 and user- k . For a Gaussian channel, $L = 1$, $\beta_{kl} = 1$ and $\gamma_{kl} = 0$, so that:

$$z_{12} = \pm \frac{A}{2N} \sum_{k=2}^K \int_{mT_b}^{(m+1)T_b} d_{1k} dt. \quad (14)$$

Bit-error-rate (or bit-error-probability) P_e can now be calculated using:

$$\begin{aligned}
P_e &= Q \left(\sqrt{\frac{E_b}{\frac{N_o}{N} + I_o}} \right) = Q \left(\sqrt{\frac{|z_{11}|^2}{\frac{N_o}{N} + |z_{12}|^2}} \right) \\
&= Q \left(\sqrt{\frac{2 \frac{|AT_b|^2}{2}}{\frac{N_o}{N} + \left| \frac{A}{2N} \sum_{k=2}^K \int_{mT_b}^{(m+1)T_b} d_{1k} dt \right|^2}} \right), \quad (15)
\end{aligned}$$

where E_b is energy-per-bit and I_o is interference term. Equation 15 shows that BER depends on the codes' cross-correlation. For a single-user system, the interference term is zero. Therefore,

$$P_e(\text{single user}) = Q \left(\sqrt{\frac{N \frac{|AT_b|^2}{2}}{N_o}} \right). \quad (16)$$

This equation shows that the performance of a single-user system depends on code-length. Since the Q -function is a monotonically decreasing function, it implies that BER decreases when N increases. For uncoded data transmission, $N = 1$. Therefore,

$$P_e^u = Q \left(\sqrt{\frac{2 \frac{|AT_b|^2}{2}}{N_o}} \right) = Q \left(\sqrt{2 \frac{E_b^u}{N_o}} \right) = Q \left(\sqrt{2(EbNo)_u} \right)$$

where E_b^u , P_e^u and $(EbNo)_u$ respectively represent energy-per-bit, bit-error-probability and bit-energy-to-noise-ratio for uncoded data transmission.

III. METHODOLOGY

To validate the mathematical model, direct software implementation of the system equations was carried out. Bit-error-rate curves and cross-correlation indices were generated from the equations. The outcome was compared with results of a simulation model. Even- and odd-degree Gold codes were generated for degree $n = 5, 6, 7, 8, 9, 10, 11$ and 12 , corresponding to $N = 31, 63, 127, 255, 511, 1023, 2047$ and 4095 respectively. Software simulations were carried out for the transmission of typically about one million random QPSK symbols through a Gaussian channel of zero mean and unit variance Gaussian noise. Following signal recovery, original transmitted data was compared with recovered data for the determination of system bit-error-rate for the various sets of codes.

IV. RESULTS AND DISCUSSION

A. Necessary condition

Equation 16 for bit-error-rate for a single user is a special case of (15). Therefore, correctness of (16) is a necessary condition for the entire mathematical model to be valid. In the treatment that follows, an attempt is hereby made to verify this.

B. Analysis versus Simulation

Here the outcome of the analysis is compared with results of software simulations for the codes. Figure 1 shows the outcome. This figure shows graphs of BER for a single user. The right-most curve on this figure is that of uncoded data transmission. The figure clearly shows close agreement between results of the analysis and simulation. The remaining curves on the figure show performance for coded data transmission for the various code lengths, for even as well as

odd-degree Gold codes. This figure shows that longer Gold codes give better error-rate performance. This is expected since process gain is a linear function of code length. On Figure 1, the graphs on the left column shows results of simulation, obtained by sending data samples (about one million data samples) into the simulation model. The graphs on the right column show results of analysis, obtained by direct implementation of the system equations of Section II. Clearly, this figure shows that the mathematical model agree closely with simulation, thereby validating the former.

C. Multiple-access Performance

Validating the mathematical model for multiple users requires a consideration of the codes' cross-correlation index (Equation 13 and 15). This is beyond the scope of this paper, and will be treated in some other communications.

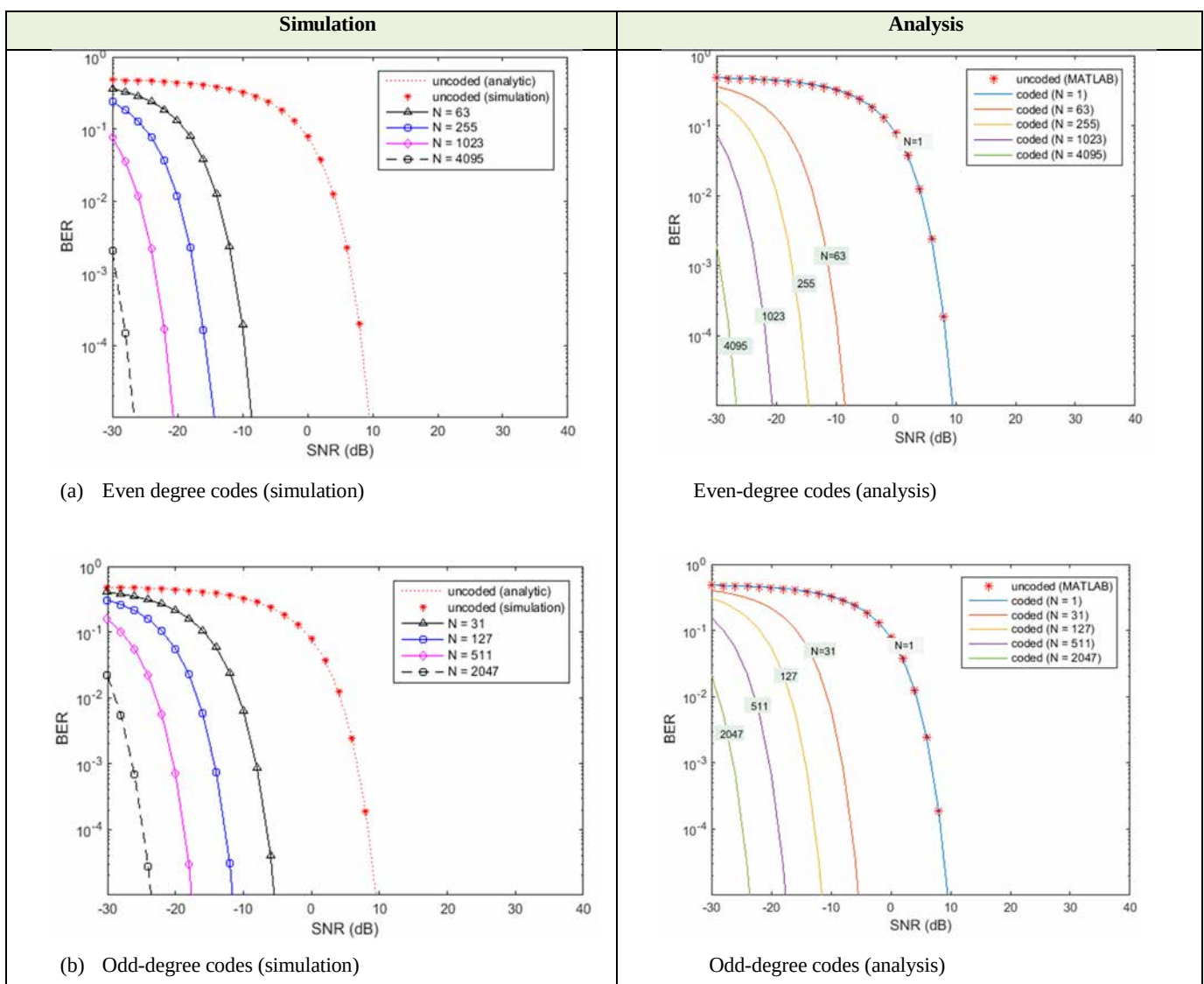


Figure 1. Comparison of results of simulation and analysis for (a) even-degree codes and (b) odd-degree codes.

V. CONCLUSION

This work presented a mathematical model for bit-error-rate of gold codes in a spread-spectrum system. The model was validated by comparing it with results of simulation carried out for the system performance, and was found to agree closely with the latter. Further work on the codes' cross-correlation index and their multiple-access performance will be presented in some other aspects of the work to be published later.

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